A Labeled Natural Deduction System for a Fragment of *CTL**

Andrea Masini* Luca Viganò** Marco Volpe***

Department of Computer Science, University of Verona, Italy
{andrea.masini | luca.vigano | marco.volpe}@univr.it

Abstract. We give a sound and complete labeled natural deduction system for an interesting fragment of CTL^*, namely the until-free version of $BCTL^*$. The logic $BCTL^*$ is obtained by referring to a more general semantics than that of CTL^*, where we only require that the set of paths in a model is closed under taking suffixes (i.e. is suffix-closed) and is closed under putting together a finite prefix of one path with the suffix of any other path beginning at the same state where the prefix ends (i.e. is fusion-closed). In other words, this logic does not enjoy the so-called limit-closure property of the standard CTL^* validity semantics.

1 Introduction

The importance of temporal logic in computer science has become clear since the seminal work of Pnueli in 1977 [9]. Interesting applications include its use as a tool for the specification and verification of programs and protocols, in the study and development of temporal databases, as a framework within which to define the semantics of temporal expressions in natural language, and as a language for encoding temporal knowledge in artificial intelligence.

Many branching temporal logics have been proposed in the literature (see [3] for a survey) varying both in the set of the operators used and in the semantics adopted. In particular, the branching-time logic CTL^* (full computation tree logic [5]) has been shown to be especially useful in developing and checking the correctness of reactive systems. In spite of its great relevance, the problem of presenting a satisfactory deduction system or even an Hilbert-style axiomatization for such a logic has been solved only recently in [12].

The aim of this work is to give a sound and complete deduction system for an interesting fragment of CTL^*, namely the until-free version of $BCTL^*$ [14]. The logic $BCTL^*$, which coincides with the logic $\forall LTFC$ described in [16], is obtained by referring to a more general semantics than that of CTL^*, where we only require that the set of paths in a model is closed under taking suffixes (i.e. is

* Partially supported by the PRIN project "CONCERTO".
** Partially supported by the PRIN project "SOFT" and the FP7-ICT-2007-1 Project no. 216471, "AVANTSSAR: Automated Validation of Trust and Security of Service-oriented Architectures" (www.avantssar.eu).
*** Partially supported by the PRIN project "SOFT".

suffix-closed) and is closed under putting together a finite prefix of one path with the suffix of any other path beginning at the same state where the prefix ends (i.e. is *fusion-closed*). In other words, this logic does not enjoy the so-called limit-closure property of the standard CTL^* validity semantics.

The until-free $BCTL^*$ logic that we consider here, to which we give the name $BCTL^*_{\_}$, restricts the set of linear temporal operators to X and G (with the usual intended meanings of "in the next time-instant" and "always in the future" respectively) and includes the universal path quantifier $\forall$, but not the until operator. As in CTL^* (and unlike CTL), we do not constrain temporal operators to be preceded by a path quantifier. From a semantic point of view, we refer to the notion of *bundled validity* (see, e.g., [12]), which will be further clarified in the following.

We give our natural deduction system for $BCTL^*_{\_}$ in the style of *labeled deduction*, a framework [6] that has been successfully employed for several non-classical, and in particular modal, logics [15,19], since labeling provides a clean and effective way of dealing with modalities and gives rise to deduction systems with good proof-theoretical properties. The basic idea is that labels allow one to explicitly encode additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to capture. So, for instance, instead of a formula A, we consider the *labeled formula* $x : A$, which intuitively means that A holds at the world denoted by x within the underlying Kripke semantics. We can also use labels to specify how worlds are related, e.g. the *relational formula* xRy states that the world y is accessible from x.

It is possible to think of a temporal logic (at least the one we consider) as a modal logic, where modal operators are used to reason on (and the accessibility relation to model) the flow of time. In the light of this consideration, we give here a labeled natural deduction system for $BCTL^*_{\_}$, where labeling allows us to formulate simple and intuitive natural deduction inference rules. We use labels to refer to possible paths rather than to time points and this view leads to a clean deduction system in which each operator (X, G, $\forall$) is seen as a modal operator and is endowed with a proper accessibility relation. Relations between the operators are expressed by means of structural rules that do not involve the operators themselves directly.

We show in this paper that our system is sound and complete, and leave for future work a detailed proof-theoretical analysis of the system (e.g. normalization), as well as the investigation of implementing automatic proof search. Moreover, we are currently working at extending the proposed approach to capture richer and more interesting logics such as full CTL^*, for which this work provides a stepping stone.

We proceed as follows. In Section 2, we give a brief presentation of the syntax and semantics and of an axiomatization of $BCTL^*_{\_}$. In Section 3, we give a labeled natural deduction system for it, which we show in Section 4 to be sound (with respect to the given semantics) and in Section 5 to be complete (with respect to the given axiomatization). We conclude, in Section 6, by comparing with related work and discussing future work.

2 The Bundled Temporal Logic $BCTL^*_-$

We introduce here the logic $BCTL^*_-$, i.e. the until-free fragment of $BCTL^*$.

2.1 Syntax

Definition 1. *Given a set $\mathcal{P}$ of propositional symbols, the set of* well-formed $BCTL^*_-$ formulas *is defined by the grammar*

$$\alpha ::= p \mid \bot \mid \alpha \supset \alpha \mid \mathsf{X}\alpha \mid \mathsf{G}\alpha \mid \forall\alpha\,,$$

where $p \in \mathcal{P}$. The set of atomic formulas *is $\mathcal{P} \cup \{\bot\}$.*

The given syntax uses a minimal set of connectives, operators, and path quantifiers. As usual, we can introduce abbreviations and use, e.g., $\neg$, $\wedge$, $\vee$ for the negation, the conjunction, and the disjunction, respectively. For instance, $\neg\alpha \equiv \alpha \supset \bot$. We can also define other temporal operators, e.g. $\mathsf{F}\alpha \equiv \neg\mathsf{G}\neg\alpha$ to express that α holds sometime in the future, and the existential path quantifier, i.e. $\exists\alpha \equiv \neg\forall\neg\alpha$.

To define a labeled deduction system for the logic $BCTL^*_-$, we extend the language with a set of labels and introduce the notions of labeled formula and relational formula. In the following, we will use the letters $b, c, d, ...$ (sometimes subscripted or superscripted) to denote labels, the symbol φ to denote a generic formula (either labeled or relational) and the symbol Γ to denote a set of formulas.

Definition 2. *Let $\mathcal{L}$ be a set of labels and let $b, c \in \mathcal{L}$. If α is a well-formed $BCTL^*_-$ formula, then $b : \alpha$ is a* labeled well-formed formula *(*labeled formula *or* lwff *for short). The set of* relational well-formed formulas *(*relational formulas *or* rwffs *for short) is defined as follows:*

$$\rho ::= b \lhd c \mid b \leqslant c \mid b \bullet c\,.$$

In the rest of the paper, we will assume given a fixed denumerable set $\mathcal{L}$ of labels. Intuitively, in our system, a label is used to refer to a path of a computation. Usually, presentations of branching time logics distinguish between

- *state formulas*, whose main operator is a boolean connective or a path quantifier and which are evaluated with respect to a state, and
- *path formulas*, whose main operator is a linear temporal operator and which are evaluated with respect to a path.

In our case, the intended meaning of an lwff $b : \alpha$ is that

- α holds in the initial state of b when α is a state formula, and that
- α holds in the path b when α is a path formula.

This will be further clarified in Section 2.2, where a semantics given only in terms of paths will be presented.

In the rwffs, we use $\lhd$, $\leqslant$ and $\bullet$ with the following intended meaning:

- $b_1 \leqslant b_2$ states that b_2 is a suffix of b_1, i.e. if $b_1 = s_1, s_2, ...$ then $b_2 = s_i, s_{i+1}, ...$ for some $i \geq 1$;
- $b_1 \lhd b_2$ states that b_2 is the maximal proper suffix of b_1, i.e. if $b_1 = s_1, s_2, s_3, ...$ then $b_2 = s_2, s_3, ...$;
- $b_1 \bullet b_2$ states that b_1 and b_2 share the same initial state, i.e. if $b_1 = s_1, s_2, s_3, ...$ and $b_2 = s'_1, s'_2, s'_3, ...$ then $s_1 = s'_1$.

2.2 Semantics

Several alternative semantics have been proposed for the branching-time logics and some equivalence results have also been showed (see, e.g., [2]). In particular, we can give two main notions of validity: the *full validity* and the *bundled validity*[1] (for a detailed account see [3,12]). If we define a *transition frame* as consisting of a set S of states and of a serial relation $\mathcal{R}$ on S, i.e. a relation such that for every s in S there exists a t in S for which $s\mathcal{R}t$ holds, then the notion of full validity is given by defining the semantics with respect to the set of all the $\mathcal{R}$-generable paths, i.e. of all the ω-sequences $s_1, s_2, ...$ such that $(s_i, s_{i+1}) \in \mathcal{R}$ for all $i \in \mathbb{N}$.

In this work, we refer instead to the notion of *bundled validity*, which determines a smaller set of valid formulas and has been used to define the subset of CTL^* called $\forall LTFC$ in [16] and $BCTL^*$ in [14]. Bundled validity is given by considering a predefined subset P of paths that is required to be (as in *bundled transition frames* of [14]):

1. *suffix-closed*, i.e. if the path $s_0, s_1, s_2...$ is in P then the path $s_1, s_2, ...$ is also in P; and
2. *fusion-closed*, i.e. if $s_1, s_2, ..., s_n, s_{n+1}, s_{n+2}, ...$ and $s'_1, s'_2, ..., s_n, s'_{n+1}, s'_{n+2}, ...$ are in P then $s_1, s_2, ..., s_n, s'_{n+1}, s'_{n+2}, ...$ is also in P.

However, here we prefer to consider a different but equivalent formulation given by frames where the basic entities (or *worlds*, in a Kripke-style terminology) are the paths of computation rather than the states. In fact, this view allows us to present a more genuine Kripke-style semantics, closer to the interpretation we want to give to the set of rules of our system.

We thus introduce $(\mathbb{N} \times \mathcal{W})$-structures [12], which are closely related to the Kamp and Ockhamist structures, described respectively in [17] and [21].

Definition 3. *A* (floored) Ockhamist frame *of countable height (in the following just* Ockhamist frame*) is a triple* $(\mathcal{T}, \prec, \simeq)$ *where:*

1. $\mathcal{T}$ *is the set of points;*
2. $\prec$ *is a transitive, anti-symmetric, irreflexive, linear relation on* $\mathcal{T}$*, i.e.:*
   (a) $\forall x, y, z.\, ((x \prec y) \wedge (y \prec z)) \Rightarrow (x \prec z)$*;*
   (b) $\forall x, y.\, \neg\, ((x \prec y) \wedge (y \prec x))$*;*

[1] An example showing that the full and the bundled validity are distinct notions is given by the formula $\alpha \equiv \forall \mathsf{G}(p \supset \exists \mathsf{X} p) \supset (p \supset \exists \mathsf{G} p)$, where p is an atomic formula. It is possible to check (see [12]) that α is valid with respect to the full semantics but not with respect to the bundled one.

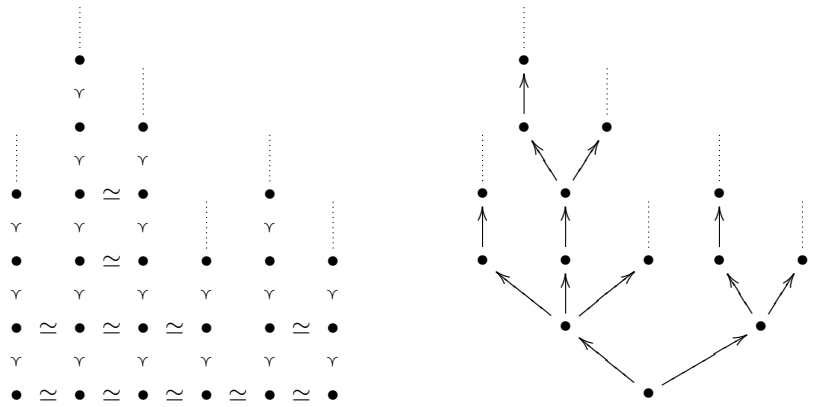


Fig. 1. An *Ockhamist frame* (left) and the corresponding *transition frame* (right)

   (c) $\forall x. \neg (x \prec x)$;
   (d) $\forall x, y, z. ((x \prec y) \wedge (x \prec z)) \Rightarrow ((z \prec y) \vee (z = y) \vee (y \prec z))$;
   (e) $\forall x, y, z. ((y \prec x) \wedge (z \prec x)) \Rightarrow ((z \prec y) \vee (z = y) \vee (y \prec z))$;
3. $\{y \mid y \prec x\}$ *is finite for each* $x \in \mathcal{T}$;
4. $\simeq$ *is an equivalence relation such that:*
   (a) if $x \simeq y$ *then it is not the case that* $x \prec y$;
   (b) if $x \simeq y$ *and* $u \prec x$ *then there is a* v *such that* $v \prec y$ *and* $u \simeq v$;
5. *there is an element* $0 \in \mathcal{T}$ *such that for each* $w \in \mathcal{T}$, *there is a* $w' \in \mathcal{T}$ *such that* $0 \simeq w'$ *and either* $w' \prec w$ *or* $w' = w$ *(the equivalence class* $0/_{\simeq}$ *is known as the* floor*).*

Intuitively, every Ockhamist point can be thought of as corresponding to a path in a transition frame and the relation $\prec$ as the equivalent of the relation "is a prefix of", i.e. $x \prec y$ stands for "the path x is a prefix of the path y". The branching nature of Ockhamist frames is hidden in the $\simeq$-equivalence relation, where the idea is that each $\simeq$-class of points contains all the paths of the corresponding transition frame that share a same initial state.

More precisely, there exists a translation [13] between Ockhamist frames and bundled transition frames (as exemplified in Fig. 1) based on the fact that Ockhamist points correspond to paths in the transition frame while points related by $\simeq$ correspond to paths with the same initial state.

In order to give a proper semantics for every linear temporal operator, we require the lines of points defined by $\prec$ to be isomorphic to the natural numbers.

Definition 4. *An Ockhamist frame* $(\mathcal{T}, \prec, \simeq)$ *is an* $(\mathbb{N} \times \mathcal{W})$-frame *iff*

1. *there is some set* $\mathcal{W}$ *such that* $\mathcal{T} = (\mathbb{N} \times \mathcal{W})$;
2. *the order* $\prec$ *is defined by* $(n, u) \prec (m, v)$ *iff* $n < m$ *and* $u = v$.

As usual, we obtain a structure by providing the frame a valuation function. In this case, we also need to require that all points in a $\simeq$-equivalence class satisfy the same set of atoms.

Definition 5. *The structure* $(\mathcal{T}, \prec, \simeq, \mathcal{V})$ *is an* $(\mathbb{N} \times \mathcal{W})$-structure *iff* $(\mathcal{T}, \prec, \simeq)$ *is an* $(\mathbb{N} \times \mathcal{W})$*-frame,* $\mathcal{V} : (\mathbb{N} \times \mathcal{W}) \to 2^{\mathcal{P}}$*, and for all* $n \in \mathbb{N}$ *and for all* $u, v \in \mathcal{W}$*, if* $(n, u) \simeq (n, v)$ *then* $\mathcal{V}(n, u) = \mathcal{V}(n, v)$*.*

It is easy to show by induction the following lemma (see [13]), which will be useful later on.

Lemma 1. *Given an* $(\mathbb{N} \times \mathcal{W})$-structure $(\mathcal{T}, \prec, \simeq, \mathcal{V})$ *and two points* (n, w) *and* (m, v) *in* $\mathcal{T}$*, if* $(n, w) \simeq (m, v)$ *then* $n = m$*.*

In order to give a semantics for our labeled system, we need to define explicitly an interpretation of labels as worlds.

Definition 6. *Given the set of labels* $\mathcal{L}$ *and an* $(\mathbb{N} \times \mathcal{W})$*-structure* $\mathcal{M} = (\mathcal{T}, \prec, \simeq, \mathcal{V})$*, where* $\mathcal{T} = (\mathbb{N} \times \mathcal{W})$ *for some set* $\mathcal{W}$*, an* interpretation *is a function* $\lambda : \mathcal{L} \to \mathcal{T}$ *that maps every label in* $\mathcal{L}$ *to a point in* $\mathcal{T}$*.*

We can now give the notion of truth directly for labeled and relational formulas. Note that truth is defined by having the temporal operators X and G operate along the $\prec$-lines of points, and the quantifier $\forall$ within a $\simeq$-equivalence class.

Definition 7. *Given an* $(\mathbb{N} \times \mathcal{W})$*-structure* $\mathcal{M} = (\mathcal{T}, \prec, \simeq, \mathcal{V})$*, where* $\mathcal{T} = (\mathbb{N} \times \mathcal{W})$ *for some set* $\mathcal{W}$*, and an interpretation* λ *on it,* truth *for an rwff or lwff* φ *is the relation* $\models^{\mathcal{M},\lambda}$ *defined as follows:*

$$
\begin{array}{lll}
\not\models^{\mathcal{M},\lambda} b : \bot; & & \\
\models^{\mathcal{M},\lambda} b_1 \lhd b_2 & \textit{iff} & \textit{there exist } n \in \mathbb{N} \textit{ and } w \in \mathcal{W} \textit{ such that} \\
 & & \lambda(b_1) = (n, w) \textit{ and } \lambda(b_2) = (n+1, w); \\
\models^{\mathcal{M},\lambda} b_1 \leqslant b_2 & \textit{iff} & \lambda(b_1) = \lambda(b_2) \textit{ or } \lambda(b_1) \prec \lambda(b_2); \\
\models^{\mathcal{M},\lambda} b_1 \bullet b_2 & \textit{iff} & \lambda(b_1) \simeq \lambda(b_2); \\
 & & \\
\models^{\mathcal{M},\lambda} b : p & \textit{iff} & p \in \mathcal{V}(\lambda(b)); \\
\models^{\mathcal{M},\lambda} b : \alpha \supset \beta & \textit{iff} & \models^{\mathcal{M},\lambda} b : \alpha \textit{ implies } \models^{\mathcal{M},\lambda} b : \beta; \\
\models^{\mathcal{M},\lambda} b : \mathsf{X}\alpha & \textit{iff} & \textit{for all } b', \models^{\mathcal{M},\lambda} b \lhd b' \textit{ implies } \models^{\mathcal{M},\lambda} b' : \alpha; \\
\models^{\mathcal{M},\lambda} b : \mathsf{G}\alpha & \textit{iff} & \textit{for all } b', \models^{\mathcal{M},\lambda} b \leqslant b' \textit{ implies } \models^{\mathcal{M},\lambda} b' : \alpha; \\
\models^{\mathcal{M},\lambda} b : \forall\alpha & \textit{iff} & \textit{for all } b', \models^{\mathcal{M},\lambda} b \bullet b' \textit{ implies } \models^{\mathcal{M},\lambda} b' : \alpha.
\end{array}
$$

When $\models^{\mathcal{M},\lambda} \varphi$*, we say that* φ *is* true *in* $\mathcal{M}$ *according to* λ*. By extension:*

$$
\begin{array}{lll}
\models^{\mathcal{M},\lambda} \Gamma & \textit{iff} & \models^{\mathcal{M},\lambda} \varphi \textit{ for all } \varphi \in \Gamma; \\
\Gamma \models^{\mathcal{M},\lambda} \varphi & \textit{iff} & \models^{\mathcal{M},\lambda} \Gamma \textit{ implies } \models^{\mathcal{M},\lambda} \varphi; \\
\models^{\mathcal{M}} \varphi & \textit{iff} & \textit{for every interpretation } \lambda, \models^{\mathcal{M},\lambda} \varphi; \\
\models^{\mathcal{M}} \Gamma & \textit{iff} & \textit{for every interpretation } \lambda, \models^{\mathcal{M},\lambda} \Gamma; \\
\Gamma \models \varphi & \textit{iff} & \textit{for every } (\mathbb{N} \times \mathcal{W})\textit{-structure } \mathcal{M} \textit{ and interpretation } \lambda, \\
 & & \Gamma \models^{\mathcal{M},\lambda} \varphi.
\end{array}
$$

We will also write $\models^{\mathcal{M},\lambda(b)} \alpha$ *for* $\models^{\mathcal{M},\lambda} b : \alpha$*, which also illustrates how truth for lwffs is related to the standard truth relation for modal and temporal logics.*

Definition 8. *We call* $BCTL^*_-$ *the set*

$$\{\alpha \mid \models^{\mathcal{M}} b : \alpha \text{ for every } b \text{ and every } (\mathbb{N} \times \mathcal{W})\text{-structure } \mathcal{M}\}.$$

The main goal of this paper is to provide a sound and complete natural deduction system for $BCTL^*_-$, which we will do in Section 3.

2.3 A Hilbert-style Axiomatization

We give now a Hilbert-style axiomatization, which we call $\mathcal{H}(BCTL^*_-)$, for the logic $BCTL^*_-$. $\mathcal{H}(BCTL^*_-)$ consists of two sets of axioms (axioms for linear temporal formulas and axioms for quantified formulas) and a set of inference rules. For the first set of axioms, we refer to a standard axiomatization for until-free *LTL* [16]:

$(L1)$ Any tautology instance	$(L2)$ $\mathsf{G}(\alpha \supset \beta) \supset (\mathsf{G}\alpha \supset \mathsf{G}\beta)$
$(L3)$ $(\mathsf{X}\neg\alpha \supset \neg\mathsf{X}\alpha) \wedge (\neg\mathsf{X}\alpha \supset \mathsf{X}\neg\alpha)$	$(L4)$ $\mathsf{X}(\alpha \supset \beta) \supset (\mathsf{X}\alpha \supset \mathsf{X}\beta)$
$(L5)$ $\mathsf{G}\alpha \supset \alpha \wedge \mathsf{XG}\alpha$	$(L6)$ $\mathsf{G}(\alpha \supset \mathsf{X}\alpha) \supset (\alpha \supset \mathsf{G}\alpha)$

The second set of axioms ensures that the path modality $\forall$ behaves as a $\Box$ in the modal logic *S5* and defines some interactions between the linear temporal operators and the path quantifier. This set of axioms comes from [12] and is slightly different from (but clearly equivalent to) the one in [16]:

$(K_\forall)$ $\forall(\alpha \supset \beta) \supset (\forall\alpha \supset \forall\beta)$ $(\forall 1)$ $\forall\alpha \supset \forall\forall\alpha$ $(\forall 2)$ $\forall\alpha \supset \alpha$ $(\forall 3)$ $\alpha \supset \forall\exists\alpha$
$(Atom)$ $p \supset \forall p$ for each atomic proposition p $(Fusion)$ $\forall\mathsf{X}\alpha \supset \mathsf{X}\forall\alpha$

Finally, we have the inference rules of modus ponens and temporal and path generalization:

(MP) If α and $\alpha \supset \beta$ then β
(Nec_X) If α then $\mathsf{X}\alpha$
(Nec_G) If α then $\mathsf{G}\alpha$
$(Nec_\forall)$ If α then $\forall\alpha$

Soundness and completeness of this axiomatization can be easily verified by adapting analogous proofs for similar axiom systems, as in the following lemma.

Lemma 2. *The axiom system* $\mathcal{H}(BCTL^*_-)$ *is sound and complete for the logic* $BCTL^*_-$*.*

Proof. (Sketch) The proof mirrors the one given in [16] for $BCTL^*$, with respect to which our axiom system only misses the two axioms concerning the operator *until*, namely:

$(L7)$ $\alpha\mathsf{U}\beta \supset \mathsf{F}\beta$
$(L8)$ $\alpha\mathsf{U}\beta \leftrightarrow \beta \vee (\alpha \wedge \mathsf{X}(\alpha\mathsf{U}\beta))$

where we denote with $\leftrightarrow$ the double implication.[2]

$\mathcal{H}(BCTL^*_-)$ is sound as it is a subset of the axiomatization in [16] and $BCTL^*_-$ structures coincide with $BCTL^*$ structures. A proof of completeness can be easily obtained by adapting the one in [16], which consists of two parts: (i) first a Henkin-style proof is given for the LTL axiomatization, by the definition of a canonical model construction; (ii) then such a construction is extended in order to consider the system for $BCTL^*$. We can modify such a proof for our case by noticing that in (i) the axioms $(L7)$ and $(L8)$ are used along the proof only to deal with formulas containing the operator *until*. We can use the same arguments to show that the axioms $(L1)-(L6)$ form a complete axiomatization for until-free LTL (as it is done for example in [7]). It is also easy to observe that the arguments in (ii) do not make use of the axioms $(L7)$ and $(L8)$. Thus we can mirror part (ii) of the proof in [16] to extend our canonical model construction for until-free LTL to a canonical model construction for $BCTL^*_-$. The main idea here is to consider the equivalence relation between points of the linear canonical model that satisfy the same state formulas and take such equivalence classes as the points of the branching canonical model. $\dashv$

3 The System $\mathcal{N}(BCTL^*_-)$

The only known deduction system for CTL^* is the Hilbert-style axiomatization given in [12]. However, it is a non-standard automata-based axiomatization, which makes use of "an unusual and unorthodox rule of inference" (as stated by Reynolds himself in [14]). Furthermore, if one is interested in a meta-theoretical and proof-theoretical analysis, Hilbert-style axiomatizations are not of a great help. Natural deduction systems have a richer syntactic structure that can be exploited to get interesting meta and proof-theoretical results. In particular, as remarked in the introduction, labeled natural deduction fits in well with the context of modal and temporal logics, by encoding into the syntax semantical properties of such logics.

In this section, we give a labeled natural deduction system, which we call $\mathcal{N}(BCTL^*_-)$, for an interesting fragment of CTL^*, the logic $BCTL^*_-$ described in Section 2. As we observed above, $\mathcal{N}(BCTL^*_-)$ provides a stepping stone for the formalization of a similar system for CTL^* that we are currently working on.

We remark that in the system $\mathcal{N}(BCTL^*_-)$ we do not make use of a proper relational labeling algebra (as, e.g., in [19]) that contains rules that derive rwffs from other rwffs or even lwffs. Since we are mainly interested in the derivation of logical formulas, we rather follow an approach that aims at simplifying the system: we use relational formulas only as side-conditions for the derivation of labeled formulas (as in Simpson's system for intuitionistic modal logic [15]) and thus in $\mathcal{N}(BCTL^*_-)$ there are no rules whose conclusion is a relational formula.

[2] In fact, our set of axioms for the branching part slightly differs from the one in [16] but the two are clearly equivalent, as remarked in [12].

3.1 The Rules of $\mathcal{N}(BCTL^*_-)$

The rules of $\mathcal{N}(BCTL^*_-)$ are given in Fig. 2. There are six kinds of rules, which we describe in the following.

Rules for the Logical Connectives The rules for the logical connectives mirror those of other labeled natural deduction systems for modal logics [15,19]. $\supset I$ and $\supset E$ are just the labeled version of the standard [10,18] natural deduction rules for implication introduction and elimination, where the notion of *discharged/open assumption* is also standard (e.g. the formula $[b : \alpha]$ is discharged in the rule $\supset I$). The rule $\bot E$ is a labeled version of *reductio ad absurdum*, where we do not enforce Prawitz's side condition that $\alpha \neq \bot$ and we do not constrain the world (b_2) in which we derive a contradiction to be the same (b_1) as in the assumption.

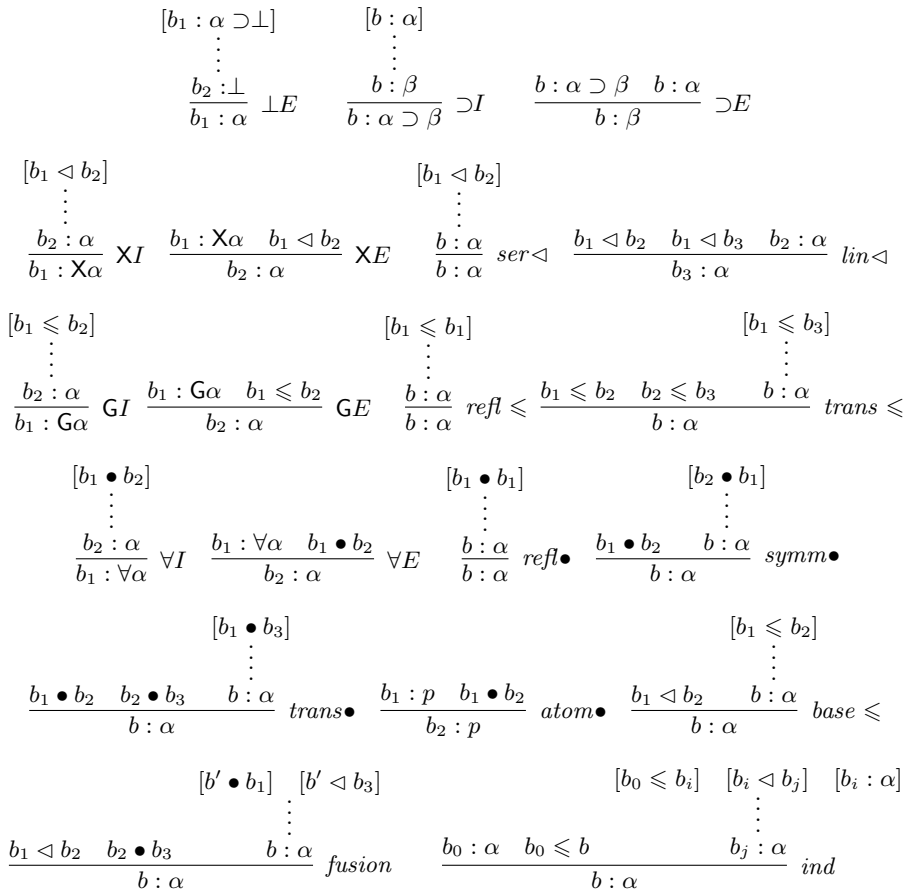
Rules for the Temporal Operators and the Path Quantifier The rules for the introduction and the elimination of X, G and $\forall$ share the same structure since they all have a "universal" formulation. In fact, let $\Box$ be one of X, G, $\forall$ and let R respectively be one of $\lhd$, $\leqslant$, $\bullet$; the idea is that the meaning of $b_1 : \Box\alpha$ is given by the metalevel implication $b_1 R b_2 \Longrightarrow b_2 : \alpha$ for an arbitrary b_2 R-accessible from b_1 (where the arbitrariness of b_2 is ensured by the side-condition on the introduction rules for X, G and $\forall$).

Rules for $\lhd$ The rule $ser\lhd$ models the fact that every world has an immediate successor and thus ensures that the suffix-closure property (as described in Section 2.2) is satisfied. The rule $lin\lhd$ specifies that such a successor must be unique.

Rules for $\leqslant$ We recall that $b_1 \leqslant b_2$ intuitively means that b_2 is a suffix of b_1. In terms of the given semantics, $\leqslant$ denotes in the syntax the reflexive and transitive closure of $\prec$ (see Definition 7). The rules $refl\leqslant$ and $trans\leqslant$ state respectively the reflexivity and transitivity of $\leqslant$.

Rules for $\bullet$ We recall from Section 2.2 that the symbol $\bullet$ in the syntax corresponds to the accessibility relation $\simeq$ in the semantics. $\simeq$ is defined as an equivalence relation and thus we have the rules $refl\bullet$, $symm\bullet$ and $trans\bullet$ that express reflexivity, symmetry and transitivity of $\bullet$ respectively. It follows that $\forall$ behaves as the modal operator $\Box$ does in the modal logic $S5$.

Finally, $atom\bullet$ mirrors the property of $(\mathbb{N} \times \mathcal{W})$-structures according to which if $x \simeq y$ then $\mathcal{V}(x) = \mathcal{V}(y)$ (see Definition 5). Intuitively, with regard to transition structures, it models the idea that two paths having the same initial state must satisfy the same set of atomic propositions and is the equivalent of the axiom $(Atom)$ in Section 2.3.



In $\mathsf{X}I$, b_2 is *fresh*, i.e. it is different from b_1 and does not occur in any assumption on which $b_2 : \alpha$ depends other than the discarded assumption $b_1 \lhd b_2$.
In $ser\lhd$, b_2 is fresh, i.e. it is different from b and does not occur in any assumption on which $b : \alpha$ depends other than the discarded assumption $b_1 \lhd b_2$.
In $\mathsf{G}I$, b_2 is fresh, i.e. it is different from b_1 and does not occur in any assumption on which $b_2 : \alpha$ depends other than the discarded assumption $b_1 \leqslant b_2$.
In $\forall I$, b_2 is fresh, i.e. it is different from b_1 and does not occur in any assumption on which $b_2 : \alpha$ depends other than the discarded assumption $b_1 \bullet b_2$.
In $atom\bullet$, p is an atomic proposition.
In *fusion*, b' is fresh, i.e. it is different from b, b_1, b_2 and b_3, and does not occur in any assumption on which $b : \alpha$ depends other than the discarded assumptions $b' \bullet b_1$ and $b' \lhd b_3$.
In *ind*, b_i and b_j are fresh, i.e. they are different from each other and from b and b_0, and do not occur in any assumption on which $b : \alpha$ depends other than the discarded assumptions of the rule.

Fig. 2. The rules of $\mathcal{N}(BCTL^*_-)$

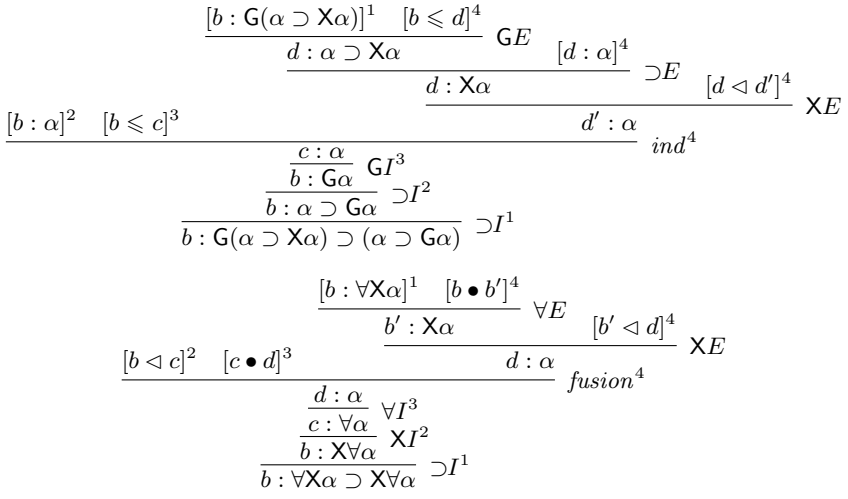


Fig. 3. Proofs of the $\mathcal{H}(BCTL^*_-)$ axioms (*L6*) and (*Fusion*)

Rules for Relations between the Operators The rule *base* $\leqslant$ expresses the fact that the relation corresponding to $\leqslant$ contains the relation corresponding to $\lhd$: in the "path terminology", it says that every path b is a prefix of its maximal proper suffix.

The rule *fusion* strictly corresponds to the *fusion-closure* property (see Section 2.2) of bundled transition frames, according to which the set of paths must be closed under putting together a finite prefix of one path with the suffix of any other path such that the prefix ends at the same state as the suffix begins. In terms of the given semantics, it roughly corresponds to condition 4(b) in the definition of an Ockhamist frame (Definition 3). In terms of the axiomatization $\mathcal{H}(BCTL^*_-)$ in Section 2.3, it is the equivalent of the axiom (*Fusion*).

Finally, we have a rule *ind* modeling the induction principle underlying the relation between $\lhd$ and $\leqslant$. It comes from the definition of $(\mathbb{N} \times \mathcal{W})$-frame (Definition 4), which requires the vertical lines of points to be isomorphic to the natural numbers. The rule is given only in terms of relations between labels, since we prefer (for proof-theoretical reasons) to restrict the treatment of operators in the system to the specific rules for their introduction and elimination.

3.2 Derivations

Given the rules in Fig. 2, the notion of derivation is the standard one for natural deduction systems [10,18]. We write $\Gamma \vdash_{\mathcal{N}(BCTL^*_-)} b : \alpha$ to say that there exists a derivation of $b : \alpha$ in the system $\mathcal{N}(BCTL^*_-)$ whose open assumptions are all contained in the set of formulas Γ. A derivation of $b : \alpha$ in $\mathcal{N}(BCTL^*_-)$ where all the assumptions are discharged is a *proof* of $b : \alpha$ in $\mathcal{N}(BCTL^*_-)$ and we then say that $b : \alpha$ is a theorem of $\mathcal{N}(BCTL^*_-)$ (and write $\vdash_{\mathcal{N}(BCTL^*_-)} b : \alpha$).

As notation, we write

$$\begin{array}{c} \varphi_1 \dots \varphi_n \\ \pi \\ b : \alpha \end{array}$$

to denote that π is a derivation of $b : \alpha$ whose set of assumptions may contain the formulas $\varphi_1, \dots, \varphi_n$.

As concrete examples, Fig. 3 contains the proofs of the $\mathcal{H}(BCTL^*_-)$ axioms $(L6)$ and $(Fusion)$.

4 Soundness

Theorem 1. *For every set Γ of labeled and relational formulas and every labeled formula $b : \alpha$, it holds that*

$$\Gamma \vdash_{\mathcal{N}(BCTL^*_-)} b : \alpha \quad \Rightarrow \quad \Gamma \models b : \alpha \, .$$

The proof proceeds by induction on the structure of the derivation of $b : \alpha$. The base case is when $b : \alpha \in \Gamma$ and is trivial. There is one step case for every rule and we show only five representative cases.

Consider an application of the rule $\mathsf{X}I$:

$$\dfrac{\begin{array}{c}[b \lhd b'] \\ \pi \\ b' : \alpha \end{array}}{b : \mathsf{X}\alpha}\ \mathsf{X}I$$

where π is a proof of $b' : \alpha$ from hypotheses in Γ', with b' fresh and with $\Gamma' = \Gamma \cup \{b \lhd b'\}$. By the induction hypothesis, for all interpretations λ, if $\models^{\mathcal{M},\lambda} \Gamma'$ then $\models^{\mathcal{M},\lambda} b' : \alpha$. We let λ be any interpretation such that $\models^{\mathcal{M},\lambda} \Gamma$, and show that $\models^{\mathcal{M},\lambda} b : \mathsf{X}\alpha$. Let (n, w) be any point such that $\lambda(b) = (n, w)$. Since λ can be trivially extended to another interpretation (still called λ for simplicity) by setting $\lambda(b') = (n + 1, w)$, the induction hypothesis yields $\models^{\mathcal{M},\lambda} b' : \alpha$, i.e. $\models^{\mathcal{M},(n+1,w)} \alpha$, and thus $\models^{\mathcal{M},\lambda} b : \mathsf{X}\alpha$.

Consider an application of the rule $\forall I$:

$$\dfrac{\begin{array}{c}[b \bullet b'] \\ \pi \\ b' : \alpha \end{array}}{b : \forall\alpha}\ \forall I$$

where π is a proof of $b' : \alpha$ from hypotheses in Γ', with b' fresh and with $\Gamma' = \Gamma \cup \{b \bullet b'\}$. By the induction hypothesis, for all interpretations λ, if $\models^{\mathcal{M},\lambda} \Gamma'$ then $\models^{\mathcal{M},\lambda} b' : \alpha$. We let λ be any interpretation such that $\models^{\mathcal{M},\lambda} \Gamma$, and show that $\models^{\mathcal{M},\lambda} b : \forall\alpha$. Let (n, w) be any point such that $\lambda(b) = (n, w)$. Now let us consider an arbitrary point (n, w') for some w'. Since λ can be trivially extended to another interpretation (still called λ for simplicity) by setting $\lambda(b') = (n, w')$, the induction hypothesis yields $\models^{\mathcal{M},\lambda} b' : \alpha$, i.e. $\models^{\mathcal{M},(n,w')} \alpha$. Given that w' is arbitrary we can conclude $\models^{\mathcal{M},\lambda} b : \forall\alpha$.

Consider the case in which the last rule applied is $\mathsf{G}E$:

$$\dfrac{\overset{\pi}{b' : \mathsf{G}\alpha} \quad b' \leqslant b}{b : \alpha}\ \mathsf{G}E$$

where π is a proof of $b' : \mathsf{G}\alpha$ from hypotheses in Γ_1, with $\Gamma = \Gamma_1 \cup \{b' \leqslant b\}$ for some set Γ_1 of formulas. By applying the induction hypothesis on π, we have:

$$\Gamma_1 \models b' : \mathsf{G}\alpha\ .$$

We proceed by considering a generic $(\mathbb{N} \times \mathcal{W})$-structure $\mathcal{M} = (\mathcal{T}, \prec, \simeq, \mathcal{V})$ and a generic interpretation λ on it such that $\models^{\mathcal{M},\lambda} \Gamma$ and showing that this entails

$$\models^{\mathcal{M},\lambda} b : \alpha\ .$$

Since $\Gamma_1 \subset \Gamma$, from the induction hypothesis we deduce $\models^{\mathcal{M},\lambda} b' : \mathsf{G}\alpha$. Furthermore $\models^{\mathcal{M},\lambda} \Gamma$ entails $\models^{\mathcal{M},\lambda} b' \leqslant b$. Then, by Definition 7, we obtain $\models^{\mathcal{M},\lambda} b : \alpha$.

Let an application of *fusion* be the last rule application in the derivation of $b : \alpha$:

$$\dfrac{b_1 \lhd b_2 \quad b_2 \bullet b_3 \qquad \begin{matrix}[b' \bullet b_1] & \overset{\pi}{} & [b' \lhd b_3] \\ & b : \alpha & \end{matrix}}{b : \alpha}\ \textit{fusion}$$

where π is a proof of $b : \alpha$ from hypotheses in Γ_2, with $\Gamma = \Gamma_1 \cup \{b_1 \lhd b_2\} \cup \{b_2 \bullet b_3\}$ and $\Gamma_2 = \Gamma_1 \cup \{b' \bullet b_1\} \cup \{b' \lhd b_3\}$ for some set Γ_1 of formulas. The side-condition ensures that b' is fresh in π. Hence, by applying the induction hypothesis on π, we have

$$\Gamma_2 \models b : \alpha\ .$$

We proceed by considering a generic $(\mathbb{N} \times \mathcal{W})$-structure $\mathcal{M} = (\mathcal{T}, \prec, \simeq, \mathcal{V})$ and a generic interpretation λ on it such that $\models^{\mathcal{M},\lambda} \Gamma$ and showing that this entails

$$\models^{\mathcal{M},\lambda} b : \alpha\ .$$

From $\models^{\mathcal{M},\lambda} \Gamma$, we deduce:

(i) there exists a point $(n, w) \in \mathcal{T}$ such that $\lambda(b_1) = (n, w)$ and $\lambda(b_2) = (n + 1, w)$;
(ii) $\lambda(b_2) \simeq \lambda(b_3)$.

We know from Lemma 1 that $\lambda(b_3) = (n + 1, v)$ for some $(n + 1, v) \in \mathcal{T}$. Then by the property $4(b)$ of Ockhamist frames (Definition 3), the point (n, v) is such that $(n, v) \simeq (n, w) = \lambda(b_1)$. Now let us consider an interpretation λ' which differs from λ only for the point assigned to b', namely $\lambda' = \lambda[b' \mapsto (n, v)]$. Note that we have defined λ' in a way such that $\models^{\mathcal{M},\lambda'} b' \bullet b_1$ and $\models^{\mathcal{M},\lambda'} b' \lhd b_3$. Since b' does not occur in Γ (by the side-condition on the application of *fusion*), we have $\models^{\mathcal{M},\lambda'} \Gamma_1$ and thus also $\models^{\mathcal{M},\lambda'} \Gamma_2$. Then, by the induction hypothesis, $\models^{\mathcal{M},\lambda'} b : \alpha$. We conclude $\models^{\mathcal{M},\lambda} b : \alpha$ by observing that the side-condition $b' \neq b$ ensures $\lambda(b) = \lambda'(b)$.

Finally, consider the case in which the last rule applied is *ind*:

$$\dfrac{\overset{\pi'}{b_0 : \alpha} \quad b_0 \leqslant b \qquad\qquad [b_0 \leqslant b_i] \quad [b_i \lhd b_j] \quad [b_i : \alpha] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\pi}{b_j : \alpha}}{b : \alpha}\ ind$$

where π is a proof of $b_j : \alpha$ from hypotheses in Γ_2 and π' is a proof of $b_0 : \alpha$ from hypotheses in Γ_1, with $\Gamma = \Gamma_1 \cup \{b_0 \leqslant b\}$ and $\Gamma_2 = \Gamma_1 \cup \{b_0 \leqslant b_i\} \cup \{b_i \lhd b_j\} \cup \{b_i : \alpha\}$ for some set Γ_1 of formulas. The side-condition on *ind* ensures that b_i and b_j are fresh in π. Hence, by applying the induction hypothesis on π and π', we have:

$$\Gamma_2 \models b_j : \alpha \qquad \text{and} \qquad \Gamma_1 \models b_0 : \alpha\,.$$

We proceed by considering a generic $(\mathbb{N} \times \mathcal{W})$-structure $\mathcal{M} = (\mathcal{T}, \prec, \simeq, \mathcal{V})$ and a generic interpretation λ on it such that $\models^{\mathcal{M},\lambda} \Gamma$ and showing that this entails

$$\models^{\mathcal{M},\lambda} b : \alpha\,.$$

First, we note that $\Gamma_1 \subset \Gamma$ and therefore $\models^{\mathcal{M},\lambda} \Gamma$ implies $\models^{\mathcal{M},\lambda} \Gamma_1$ and, by the induction hypothesis on π', $\models^{\mathcal{M},\lambda} b_0 : \alpha$. Let $\lambda(b_0) = (n, w)$ for some $(n, w) \in \mathcal{T}$. From $\models^{\mathcal{M},\lambda} \Gamma$, we deduce $\models^{\mathcal{M},\lambda} b_0 \leqslant b$ and thus $\lambda(b) = (n + k, w)$ for some $k \in \mathbb{N}$. We show by induction on k that $\models^{\mathcal{M},\lambda} b : \alpha$. As a base case, we have $k = 0$; it follows that $\lambda(b) = \lambda(b_0)$ and thus trivially that $\models^{\mathcal{M},\lambda} b_0 : \alpha$ entails $\models^{\mathcal{M},\lambda} b : \alpha$. Let us consider now the induction step. Given a label b_{k-1} such that $\lambda(b_{k-1}) = (n + k - 1, w)$, we show that the induction hypothesis $\models^{\mathcal{M},\lambda} b_{k-1} : \alpha$ entails the thesis $\models^{\mathcal{M},\lambda} b : \alpha$. We can build an interpretation λ' that differs from λ only in the points assigned to b_i and b_j, namely $\lambda' = \lambda[b_i \mapsto (n+k-1, w)][b_j \mapsto (n+k, w)]$. It is easy to verify that the interpretation λ' is such that the following three conditions hold:

(i) $\models^{\mathcal{M},\lambda'} b_i : \alpha$;
(ii) $\models^{\mathcal{M},\lambda'} b_0 \leqslant b_i$;
(iii) $\models^{\mathcal{M},\lambda'} b_i \lhd b_j$.

Furthermore, the side-condition on the rule *ind* ensures that λ and λ' agree on all the labels occurring in Γ_1, from which we can infer that also $\models^{\mathcal{M},\lambda'} \Gamma_1$ must hold. It follows that $\models^{\mathcal{M},\lambda'} \Gamma_2$ and thus, by the induction hypothesis on π, that $\models^{\mathcal{M},\lambda'} b_j : \alpha$. We conclude $\models^{\mathcal{M},\lambda} b : \alpha$ by observing that $\lambda'(b_j) = \lambda(b)$.

5 Completeness

The proposed natural deduction system $\mathcal{N}(BCTL^*_-)$ consists of only finitary rules; consequently, it cannot be strongly complete.[3] In fact, it is easy to check that $\{b : \mathsf{X}^i\alpha\}_{i<\omega} \models b : \mathsf{G}\alpha$ but (via soundness) we can see that $\{b : \mathsf{X}^i\alpha\}_{i<\omega} \nvdash b : \mathsf{G}\alpha$, where $\mathsf{X}^0\alpha$ is just α and $\mathsf{X}^{i+1}\alpha$ stands for $\mathsf{X}\mathsf{X}^i\alpha$. Nevertheless, our system $\mathcal{N}(BCTL^*_-)$ is weakly complete with respect to $BCTL^*_-$, namely:

[3] This is not a problem of our formulation: all the finitary deduction systems for temporal logics equipped with at least the operators X and G have such a defect (see, e.g., [8, Chapter 6]).

Theorem 2. *For every labeled formula $b : \alpha$ it holds:*

$$\models b : \alpha \quad \Rightarrow \quad \vdash_{\mathcal{N}(BCTL^*_-)} b : \alpha \, .$$

The most "economic" way to prove the theorem is to show that $\mathcal{N}(BCTL^*_-)$ is complete with respect to the axiomatization $\mathcal{H}(BCTL^*_-)$ given in Section 2.3, which is sound and complete for the logic $BCTL^*_-$. That is, we need to prove (i) that every axiom of $\mathcal{H}(BCTL^*_-)$ is provable in $\mathcal{N}(BCTL^*_-)$ and (ii) that $\mathcal{N}(BCTL^*_-)$ is closed under the (labeled equivalent of the) rules of inference of $\mathcal{H}(BCTL^*_-)$. Showing (ii) is straightforward and we omit it here. As for (i), we have already given the proofs of the $\mathcal{H}(BCTL^*_-)$ axioms $(L6)$ and $(Fusion)$ in Fig. 3. As a further example, we can prove axiom $(L5)$ as follows

$$\dfrac{\dfrac{[b : \mathsf{G}\alpha]^1 \quad [b \leqslant b]^2}{b : \alpha}\, \mathit{refl}^{\leqslant 2}}{b : \alpha}\, \mathsf{G}E \qquad \dfrac{\dfrac{[b \lhd c]^3 \quad \dfrac{\dfrac{[b \leqslant c]^5 \quad [c \leqslant d]^4}{d : \alpha}\, \mathit{trans}^{\leqslant 6} \qquad \dfrac{[b : \mathsf{G}\alpha]^1 \quad [b \leqslant d]^6}{d : \alpha}\, \mathsf{G}E}{d : \alpha}\, \mathit{base}^{\leqslant 5}}{c : \mathsf{G}\alpha}\, \mathsf{G}I^4}{b : \mathsf{XG}\alpha}\, \mathsf{X}I^3 \qquad \dfrac{b : \alpha \wedge \mathsf{XG}\alpha}{b : \mathsf{G}\alpha \supset (\alpha \wedge \mathsf{XG}\alpha)}\, \supset I^1$$

where, for simplicity, we have employed the rule $\wedge I$ for conjunction introduction, which is derived from the other propositional rules as is standard:

$$\dfrac{b : \alpha_1 \quad b : \alpha_2}{b : \alpha_1 \wedge \alpha_2}\, \wedge I \qquad \text{abbreviates} \qquad \dfrac{\dfrac{\dfrac{[b : \alpha_1 \supset (\alpha_2 \supset \bot)]^1 \quad b : \alpha_1}{b : \alpha_2 \supset \bot}\, \supset E \qquad b : \alpha_2}{b : \bot}\, \supset E}{b : (\alpha_1 \supset (\alpha_2 \supset \bot)) \supset \bot}\, \supset I^1$$

6 Conclusions

We have given a labeled natural deduction system for a fragment of CTL^* — $BCTL^*$ without *until* — and shown that it is sound and complete.

We have already considered some relevant related works in the previous sections. Other labeled natural deduction systems for branching time logics have been proposed, e.g. [1] and [11] both give labeled natural deduction systems for CTL. The main distinctive feature of our system is that reasoning only in terms of paths gives us the possibility of considering also the path quantifier $\forall$ as a modal operator and thus of getting a labeled system as clean as the ones for other modal logics [15,19].

In [14], a tableau-based decision procedure for $BCTL^*$ is given. The tableau construction differs from the traditional tree-shaped one and consists, like for other tableau systems for temporal logics, e.g. [4,20], in starting with a graph and iteratively pruning away some nodes until a success or a failure condition is reached. We remark that the focus of our work, instead, mainly concerns the definition of a deduction system with good proof-theoretical properties.

In fact, we are currently working on normalization for our system $\mathcal{N}(BCTL^*_-)$, where the main difficulties arise, as in deduction systems for Peano Arithmetics and as expectable, from the presence of a temporal induction principle. Further current work is oriented towards automated reasoning and towards extensions of the system in order to capture richer logics such as CTL^*.

References

1. Bolotov, A., Grigoriev, O., Shangin, V.: Natural Deduction Calculus for Computation Tree Logic. In: Proceedings of the John Vincent Atanasoff Symposium, pp. 175–183 (2006)
2. Emerson, E. A.: Alternative Semantics for Temporal Logics. Theoretical Computer Science 26, 121–130 (1983)
3. Emerson, E. A.: Temporal and Modal Logic. In: Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics (B), pp. 995–1072 (1990)
4. Emerson, E. A., Halpern, J. Y.: Decision Procedures and Expressiveness in the Temporal Logic of Branching Time. Journal of Computer and System Sciences. 30(1), 1–24 (1985)
5. Emerson, E. A., Sistla, A. P.: Deciding Full Branching Time Logic. Information and Control 61(3), 175–201 (1984)
6. Gabbay, D.: Labelled Deductive Systems. Clarendon Press (1996)
7. Gabbay, D., Pnueli, A., Shelah, S., Stavi, J.: On the Temporal Analysis of Fairness. In: Proceedings of POPL'80, pp. 163–173 (1980)
8. Kröger, F.: Temporal Logic of Programs. Springer-Verlag (1987)
9. Pnueli, A.: The Temporal Logic of Programs. In: Proceedings of FOCS 18, pp. 46–57 (1977)
10. Prawitz, D.: Natural Deduction. Almqvist and Wiksell (1965)
11. Renteria, C., Haeusler, E.: A Natural Deduction System for CTL. Bulletin of the Section of Logic 31(4), 231–240 (2002)
12. Reynolds, M.: An Axiomatization of Full Computation Tree Logic. Journal of Symbolic Logic 66(3), 1011–1057 (2001)
13. Reynolds, M.: An Axiomatization of PCTL*. Information and Computation 201(1), 72–119 (2005)
14. Reynolds, M.: A Tableau for Bundled CTL*. Journal of Logic and Computation 17(1), 117–132 (2007)
15. Simpson, A.: The Proof Theory and Semantics of Intuitionistic Modal Logic. PhD thesis, College of Science and Engineering, School of Informatics, University of Edinburgh (1994)
16. Stirling, C.: Modal and Temporal Logics. In: Handbook of Logic in Computer Science, Volume 2, pp. 477–563. Oxford University Press (1992)
17. Thomason, R. H.: Combinations of Tense and Modality. In: Handbook of Philosophical Logic: Extensions of Classical Logic, pp. 135–165. Reidel (1984)
18. Troelstra, A., Schwichtenberg, H.: Basic Proof Theory. Cambridge University Press (2000)
19. Viganò, L.: Labelled Non-Classical Logics. Kluwer Academic Publishers (2000)
20. Wolper, P.: The Tableau Method for Temporal Logic: An Overview. Logique et Analyse 110, 119–136 (1985)
21. Zanardo, A.: Branching-Time Logic with Quantification over Branches: The Point of View of Modal Logic. Journal of Symbolic Logic 61(1), 1–39 (1996)